Nonlinear composition operators in generalized Morrey spaces

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First of all I would like to thank the organizers for giving me the opportunity to talk here today.

Today's talk concerns the regularity properties of the composition operator T_f associated to a function

$$f : \mathbb{R} \to \mathbb{R}$$

and defined by

$$\begin{array}{rrl} T_f : & \mathbb{R}^\Omega \to & \mathbb{R}^\Omega \\ & g \mapsto & f \circ g \end{array}$$

in the frame of generalized Morrey spaces in a non-empty open subset Ω of $\mathbb{R}^n$.

We ask for which Borel measurable functions f the map T_f

maps a generalized Morrey space to itself,

is continuous, uniformly continuous, α-Hölder continuous, Lipschitz continuous.

in a generalized Morrey space of functions in Ω.

For extensive references on nonlinear composition operators, we refer to the monographs of

J. Appell and P.P. Zabreiko. (1990) Nonlinear Superposition Operators. Cambridge University Press, Cambridge.

T. Runst and W. Sickel, *Sobolev Spaces of Fractional order, Nemytskij Operators*, and Nonlinear Partial Differential Equations, De Gruyter, Berlin (1996).

R.M. Dudley and R. Norvaiša, *Concrete functional calculus*. Springer Monographs in Mathematics. Springer, New York, 2011.

Today we will NOT talk about the Koopman composition operator

$$\begin{aligned} C_g : \quad \mathbb{R}^\Omega &\to \quad \mathbb{R}^\Omega \\ f &\mapsto \quad f \circ g \end{aligned}$$

for some $g : \Omega \to \Omega$ as in

N. Hatano, M. Ikeda, I. Ishikawa, and Y. Sawano, *Boundedness of composition operators on Morrey spaces and weak Morrey spaces*, arXiv:2008.12464v1 (2020)

We recall the definition of generalized Morrey space:

$\mathbb{B}_n(x,r) = \{y \in \mathbb{R}^n : \ |x-y| < r\}$

Ω an open subset of $\mathbb{R}^n$

$M(\Omega)$ = set of measurable functions from Ω to $\mathbb{R}$

w : $]0,+\infty[\to [0,+\infty[$ a 'weight function'

$p \in [1,+\infty[$

If $g : \ \Omega \to \mathbb{R}$ is measurable, $\rho \in]0,+\infty]$, we set

$|g|_{\rho,w,p,\Omega} \equiv \sup_{(x,r)\in\Omega\times]0,\rho[} w(r) \|g\|_{L_p(\mathbb{B}_n(x,r)\cap\Omega)}$

The generalized Morrey space with weight w and exponent p is the space

$\mathcal{M}_p^w(\Omega) \equiv \left\{ g \in M(\Omega) : \ |g|_{+\infty,w,p,\Omega} < +\infty \right\}$

with the norm

$\|g\|_{\mathcal{M}_p^w(\Omega)} \equiv |g|_{+\infty,w,p,\Omega} \qquad \forall g \in \mathcal{M}_p^w(\Omega)$

The classical weights for $0 < \lambda < n/p$:

$r^{-\lambda} \quad \forall r \in]0, +\infty[\,,$

$$w_{\lambda,1}(r) \equiv \begin{cases} r^{-\lambda} & \text{if } r \in]0,1[\,, \\ 0 & \text{if } r \in [1,+\infty[\,, \end{cases}$$

$$w_{\lambda}(r) \equiv \begin{cases} r^{-\lambda}, & \forall r \in]0,1[\\ 1 & \forall r \in [1,+\infty[. \end{cases}$$

$\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ is the classical homogeneous Morrey space of exponents λ and p,

$\mathcal{M}_p^{w_{\lambda,1}}(\Omega)$ is the classical inhomogeneous Morrey space of exponents λ and p

and one can prove that

$$M_p^{\lambda}(\Omega) \equiv \mathcal{M}_p^{w_\lambda}(\Omega) = \mathcal{M}_p^{w_{\lambda,1}}(\Omega) \cap L_p(\Omega)\,,$$

where $\mathcal{M}_p^{w_{\lambda,1}}(\Omega) \cap L_p(\Omega)$ is endowed with the maximum of the norms of

$\mathcal{M}_p^{w_{\lambda,1}}(\Omega)$ and $L_p(\Omega)$.

The vanishing generalized Morrey space with weight w and exponent p is the subspace

$$\mathcal{M}_p^{w,0}(\Omega) \equiv \left\{ g \in \mathcal{M}_p^w(\Omega) : \lim_{\rho \to 0} |g|_{\rho,w,p,\Omega} = 0 \right\}$$

of $\mathcal{M}_p^w(\Omega)$.

The subspace $\mathcal{M}_p^{w,0}(\Omega)$ is well known to be closed in $\mathcal{M}_p^w(\Omega)$.

Our assumptions on the weight $w :]0, +\infty[\to [0, +\infty[$

• w is not identically equal to 0

• w is decreasing

• $\lim_{r\to 0} w(r)r^{n/p} = 0$

• there exists $\rho_0 \in]0, 1]$ such that

$w(r)(r)^{n/p}$ is continuous and increasing for $r \in]0, \rho_0[$

$\exists c > 0$ such that $w(r) \leq cw(1/\alpha)w(\alpha r)$ $(*)$

for all $\alpha > 1/\rho_0$, $0 < r < \rho_0$ such that $\alpha r < \rho_0$

REMARK:

$(*)$ implies that we can estimate $\|g(\alpha(\cdot - x^o))\|_{\mathcal{M}_p^w(\mathbb{R}^n)}$

in terms of $w(1/\alpha)(1/\alpha)^{n/p}\|g\|_{\mathcal{M}_p^w(\mathbb{R}^n)}$

when g has support in a ball $\mathbb{B}_n(0, M)$

• All the above assumptions are satified by the classical weights with $0 < \lambda < n/p$, $p \in [1, +\infty[$.

• If Ω is bounded then

$$\mathcal{M}_p^{r^{-\lambda}}(\Omega) = \mathcal{M}_p^{w_{\lambda,1}}(\Omega) = \mathcal{M}_p^{w_\lambda}(\Omega)$$

with equivalent norms

• We are not interested into 'limiting cases' in which the Morrey space equals either a Lebesgue space or $\{0\}$.

The analysis of T_f in Lebesgue spaces depends on whether

$$m_n(\Omega) < +\infty \qquad \text{or} \qquad m_n(\Omega) = +\infty$$

Here for generalized Morrey spaces the analysis depends on whether

$$1 \in \mathcal{M}_p^w(\Omega) \qquad \text{or} \qquad 1 \notin \mathcal{M}_p^w(\Omega)$$

and for vanishing generalized Morrey spaces on whether

$$1 \in \mathcal{M}_p^{w,0}(\Omega) \qquad \text{or} \qquad 1 \notin \mathcal{M}_p^{w,0}(\Omega)$$

Under our assumptions on w:

$$1 \in \mathcal{M}_p^w(\Omega) \Rightarrow 1 \in \mathcal{M}_p^{w,0}(\Omega)$$

So the two of them coincide

Remark:

If $m_n(\Omega) < +\infty$, then $1 \in \mathcal{M}_p^w(\Omega)$, and under our assumptions we also have $1 \in \mathcal{M}_p^{w,0}(\Omega)$

If $\eta_w \equiv \inf_{r\in]0,+\infty[} w(r) > 0$, then

$1 \in \mathcal{M}_p^w(\Omega) \Rightarrow m_n(\Omega) < +\infty$

However the only classical weight for which $\eta_w > 0$ is

$$w_\lambda(r) \equiv \begin{cases} r^{-\lambda}, & \forall r \in]0,1[\\ 1 & \forall r \in [1,+\infty[. \end{cases}$$

but not in general:

$1 \in \mathcal{M}_p^{w_{\lambda,1}}(\mathbb{R}^n)$ for all $p \in [1,+\infty[$, $\lambda \in]0,n/p[$.

There are cases in which $m_n(\Omega) = +\infty$ and

$1 \notin \mathcal{M}_p^w(\Omega)$.

So for example $1 \notin \mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ for all $p \in [1,+\infty[$, $\lambda \in]0,n/p[$.

The **'action problem'** of T_f:

• characterize those Borel functions $f : \mathbb{R} \to \mathbb{R}$ such that

$f \circ g \in \mathcal{M}_p^w(\Omega)$ for all $g \in \mathcal{M}_p^w(\Omega)$

i.e., such that $T_f[\mathcal{M}_p^w(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$

• characterize those Borel functions $f : \mathbb{R} \to \mathbb{R}$ such that

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i.e., such that $T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^{w,0}(\Omega)$

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, $p \in [1, +\infty[$.

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f[\mathcal{M}_p^w(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^{w,0}(\Omega)$ if and only

there exist a, $b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$, i.e. f is sub-affine

• If $1 \notin \mathcal{M}_p^w(\Omega)$, then we have

$T_f[\mathcal{M}_p^w(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^w(\Omega)$ if and only if

$T_f[\mathcal{M}_p^{w,0}(\Omega)] \subseteq \mathcal{M}_p^{w,0}(\Omega)$ if and only

there exists $a \in [0, +\infty[$ such that

$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$, i.e. f is sub-linear.

For the sufficiency: N. Kydyrmina & M. L. Eurasian Mathematical Journal, **7**, No. 2 (2016), pp. 50–67. [where Sobolev-Morrey spaces have been considered]

The proof of the necessity is based on a generalization of the proof for Lebesgue spaces of G. Bourdaud and based on a Lemma of Y. Katznelson

that says that the acting condition of T_f implies a property of boundedness of T_f

on bounded sets of g's with uniformly bounded (small) support and with small norm.

Lemma of Y. Katznelson:

$E_1 \hookrightarrow L_{1\,\mathrm{loc}}(\Omega), \quad E_2 \hookrightarrow L_{1\,\mathrm{loc}}(\Omega)$

E_1 is complete,

for each $\varphi \in \mathcal{D}(\Omega)$ we have

$g \in E_2 \Rightarrow \varphi g \in E_2$

and that the multiplication operator by φ is continuous in E_2

$f(0) = 0$ and $T_f[E_1] \subseteq E_2$

Then there exist c_1, $c_2 > 0$, $x^o \in \Omega$, $q \in]0, +\infty[$ such that

$Q_{x^o,q} \equiv x^o +]-q, q[^n \subseteq \Omega$

and

$g \in E_1, \ \operatorname{supp} g \subseteq Q_{x^o,q}, \ \|g\|_{E_1} \leq c_1$

$\Rightarrow \|f \circ g\|_{E_2} \leq c_2.$

The problem of **uniform continuity** of T_f:

• characterize those Borel functions $f : \mathbb{R} \to \mathbb{R}$ such that T_f is uniformly continuous.

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is uniformly continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

[uniformly continuous functions are always sub-affine]

And how about case $1 \notin \mathcal{M}_p^w(\Omega)$?

Here the answer is more surprizing:

If $1 \notin \mathcal{M}_p^w(\Omega)$ and if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is uniformly continuous,

then f is Lipschitz continuous and $f(0) = 0$.

On the other hand we shall see that the Lipschitz continuity of f and $f(0) = 0$ is sufficient for the Lipschitz continuity of T_f.

The problem of α-**Hölder continuity** of T_f for $\alpha \in]0,1[$:

Here the point is that

• If $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is α-Hölder continuous

then $|f|_\alpha \leq \|\chi_E\|_{\mathcal{M}_p^w(\Omega)}^{\alpha-1} |T_f|_\alpha$

for all measurable subsets E of Ω of finite nonzero measure.

In particular, f is α-Hölder continuous.

If f is not constant, then $1 \in \mathcal{M}_p^{w,0}(\Omega)$

and $|f|_\alpha \|1\|_{\mathcal{M}_p^w(\Omega)}^{1-\alpha} \leq |T_f|_\alpha$

• If f is Borel measurable but NOT constant, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is α-Hölder continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is α-Hölder continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is α-Hölder continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous and $1 \in \mathcal{M}_p^{w,0}(\Omega)$.

• If the above equivalent conditions hold,

then $|T_f|_\alpha \leq |f|_\alpha \|1\|_{\mathcal{M}_p^w(\Omega)}^{1-\alpha}$

The problem of **Lipschitz continuity** of T_f:

• If $1 \in \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is Lipschitz continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

• If $1 \notin \mathcal{M}_p^w(\Omega)$, then

$T_f : \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is Lipschitz continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is Lipschitz continuous if and only if

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and $f(0) = 0$.

The problem of **continuity** of T_f:

Here unfortunately we have only sufficient conditions and necessary conditions.

A necessary condition for continuity:

• If $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is continuous,

then f is continuous and there exist a, $b \in [0, +\infty[$ such that

$$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$$

• If $1 \notin \mathcal{M}_p^w(\Omega)$ and if $T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^w(\Omega)$ is continuous,

then f is continuous and there exists $a \in [0, +\infty[$ such that

$$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$$

A sufficient condition for continuity:

If $m_n(\Omega) < +\infty$ (a case in which $1 \in \mathcal{M}_p^{w,0}(\Omega)$) and if

$$c_f \equiv \sup \left\{ \frac{|f(x)-f(y)|}{1+|x-y|} \, : \, x, y \in \mathbb{R} \right\} < +\infty \, ,$$

then

$$T_f \, : \, \mathcal{M}_p^w(\Omega) \to \mathcal{M}_p^w(\Omega) \text{ is continuous.}$$

As shown in L & Bourdaud and Sickel (2002) condition $c_f < +\infty$ is equivalent to:

There exist a_1, $a_2 \in]0, +\infty[$ such that

$|f(x) - f(y)| \leq a_2$ for all x, $y \in \mathbb{R}$

such that $|x - y| \leq a_1$.

and is a necessary and sufficient condition for the action of T_f in $BMO(\mathbb{R}^n)$.

A sufficient condition for continuity in generalized vanishing Morrey spaces:

• If f is continuous and if there exists $a \in [0, +\infty[$ such that

$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$, then

T_f :

$$(\mathcal{M}_p^{w,0}(\Omega) \cap L_p(\Omega), \|\cdot\|_{\mathcal{M}_p^w(\Omega)\cap L_p(\Omega)}) \to \mathcal{M}_p^{w,0}(\Omega).$$

is continuous

• If $1 \in \mathcal{M}_p^{w,0}(\Omega)$, f is continuous and if there exist a, $b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$, then

T_f :

$$(\mathcal{M}_p^{w,0}(\Omega) \cap L_p(\Omega), \|\cdot\|_{\mathcal{M}_p^w(\Omega)\cap L_p(\Omega)}) \to \mathcal{M}_p^{w,0}(\Omega).$$

is continuous.

A necessary and sufficient condition for continuity in generalized vanishing Morrey spaces

under the special assumption

$\eta_w \equiv \inf_{r\in]0,+\infty[} w(r) > 0$

• If $1 \in \mathcal{M}_p^{w,0}(\Omega)$, then

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is continuous if and only if

f is continuous and there exist $a, b \in [0, +\infty[$ such that

$|f(t)| \leq a|t| + b \qquad \forall\, t \in \mathbb{R}$.

Unfortunately the only classical weight for which

$\eta_w > 0$ is $w_\lambda(r) \equiv \begin{cases} r^{-\lambda}, & \forall r \in]0,1[\\ 1 & \forall r \in [1,+\infty[. \end{cases}$

If Ω is bounded:

$$\mathcal{M}_p^{r^{-\lambda}}(\Omega) = \mathcal{M}_p^{w_{\lambda,1}}(\Omega) = \mathcal{M}_p^{w_\lambda}(\Omega)$$

with equivalent norms

under the special assumption $\eta_w > 0$:

• If $1 \notin \mathcal{M}_p^{w,0}(\Omega)$, then

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w,0}(\Omega)$ is continuous if and only if

$T_f : \mathcal{M}_p^{w,0}(\Omega) \to \mathcal{M}_p^{w}(\Omega)$ is continuous if and only if

f is continuous and there exists $a \in [0, +\infty[$ such that

$$|f(t)| \leq a|t| \qquad \forall\, t \in \mathbb{R}$$

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THANK YOU FOR YOUR ATTENTION!