Overlap Gap Property: an Algorithmic Barrier to Optimizing Over Random Structures

David (Dato) Gamarnik

MIT (2005-Present)

Tbilisi State University (1986-1990)

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XII INTERNATIONAL CONFERENCE OF THE GEORGIAN MATHEMATICAL UNION

- Survey paper with the same title in Proceedings of National Academy of Science (PNAS), 2021

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- Karp [1976] Improve half-optimality?
- Still open. This is embarrassing...

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- Nothing better known.

Statistics-to-Computation gap

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Problems exhibiting a similar *statistics-to-computation* gap:

Random K-SAT
Proper coloring of a random graphs
MaxCut on random hypergraphs
Ground state of a spin glass model
Stochastic Block Model
LDPC Codes
Planted Clique
Spiked Tensor problem
Sparse Regression and Phase Retrieval
Sparse Covariance Estimation problem
Graph alignment
Binary perceptron
Mixture of Gaussians
etc, etc.

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- Change in the geometry of solutions at the onset of hardness, **Overlap Gap Property (OGP)**
- Originating in the theory of spin glass. Giorgio Parisi (Nobel Prize Physics 2021)

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(a) Emerges in most models known to exhibit an apparent algorithmic hardness
(b) Consistent with the hardness/tractability phase transition for many models analyzed to the day
(c) Allows to mathematically rigorously rule out a large class of algorithms as potential contenders, specifically the algorithms which exhibit the input stability (noise insensitivity), such as **Boolean circuits** (this talk).

Overlap Gap Property (OGP) $\rightarrow$ Algorithmic Lower Bounds

Theorem (Informal, G, Jagannath & Wein [2022])

If polynomial size Boolean circuit $\mathcal{C}$ with depth p_n finds better than half-optimum ind set in $\mathbb{G}(n, d/n)$, then its depth is at least

$$p_n \geq \Omega\left(\frac{\log n}{\log \log n}\right).$$

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- Half-optimum ind sets can be found by depth $O(1)$ circuits.
- State of the art $o(\log n / \log \log n)$, Rossman [2010], Li, Razborov & Rossman [2017], (though for the decision not the search problem).

Overlap Gap Property (OGP) $\to$ Algorithmic Lower Bounds

Theorem (Informal, G, Jagannath & Wein [2022])

If Boolean circuit $\mathcal{C}$ with a constant depth $O(1)$ finds better than half-optimum ind set in $\mathbb{G}(n, d/n)$, then its size is at least

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- State of the art lower bound is $\exp\left(\log^{\Theta(1)} n\right)$, Rossman [2010].

OGP for Independent Sets

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Theorem (G & Sudan [2014])

Fix $\frac{1}{2} + \frac{1}{2\sqrt{2}} < \alpha < 1$. *There exists* $0 \leq \nu_1 < \nu_2 < 1$ *such that with prob* $1 - \exp(-\Omega(n))$ *for every two* α*-optimum independent sets* I, J *in* $\mathbb{G}(n, d/n)$

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This was used to rule out local (Factor of IID) algorithms.

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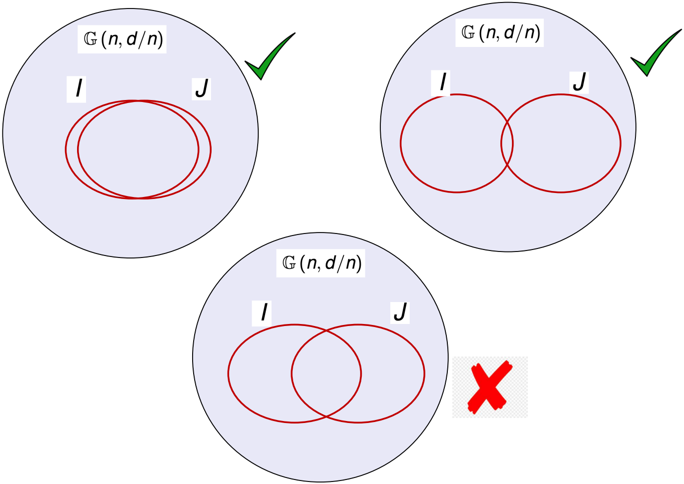
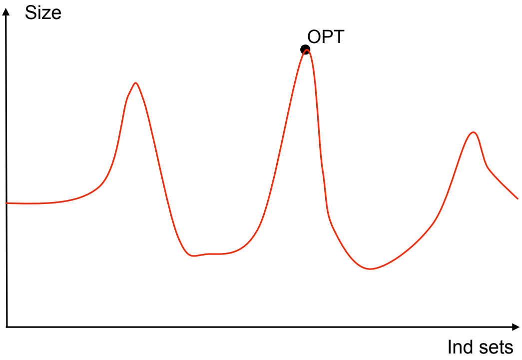


Landscape of the OGP



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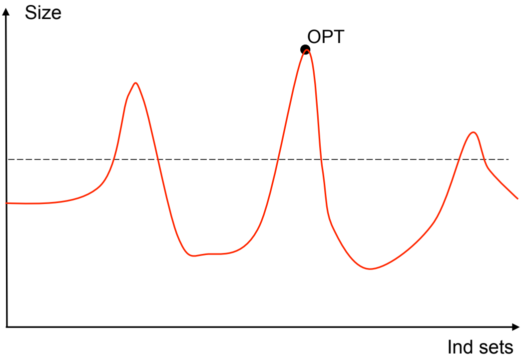

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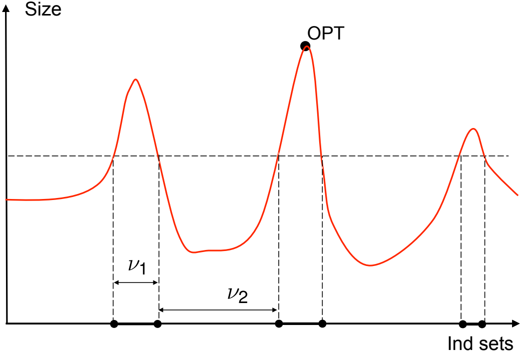
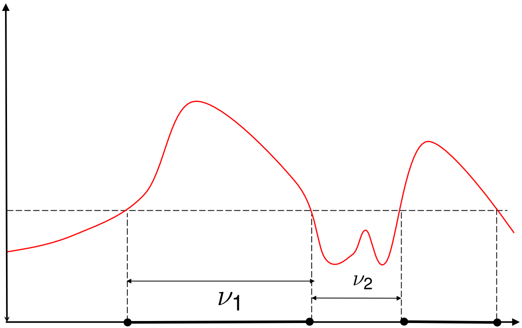


$\nu_1 < \nu_2$

Comparison with clustering



$\nu_1 > \nu_2$

Note: OGP is stronger than clustering!

References on OGP based results

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- G, Jagannath & Wein [2022] Boolean circuits (**this talk**)
- G, Jagannath & Wein [2020] Low degree polynomials, Langevin dynamics
- G & Jagannath [2020] AMP algorithms
- Coja-Oghlan, Haqshenas & Hetterich [2017] Random Walk (WAKLSAT)
- G & Sudan [2017] Survey Propagation algorithms
- Farhi, G & Gutmann [2017] Quantum (QAOA) algorithms
- G, Kizildag, Perkins & Xu [In progress] Kim-Roche algorithm for Binary perceptron
- Rahman & Virag [2017], Wein [2020]
- Bresler & Huang [2021],Huang & Sellke [2021]

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Theorem

Fix $\frac{1}{2} + \frac{1}{2\sqrt{2}} < \alpha < 1$. For all large enough d, there exists $0 \le \nu_1 < 1/2 < \nu_2 < 1$ such that with prob $1 - \exp(-\Omega(n))$ for every $0 \le t \le \binom{n}{2}$ and every α-optimum independent sets I_0 in $\mathbb{G}_0$ and I_t in $\mathbb{G}_t$

$$\frac{|I_0 \cap I_t|}{\mathcal{OPT}} \in [0, \nu_1] \cup [\nu_2, 1].$$

Furthermore, when $t = \binom{n}{2}$ only $\in [0, \nu_1]$ is possible.

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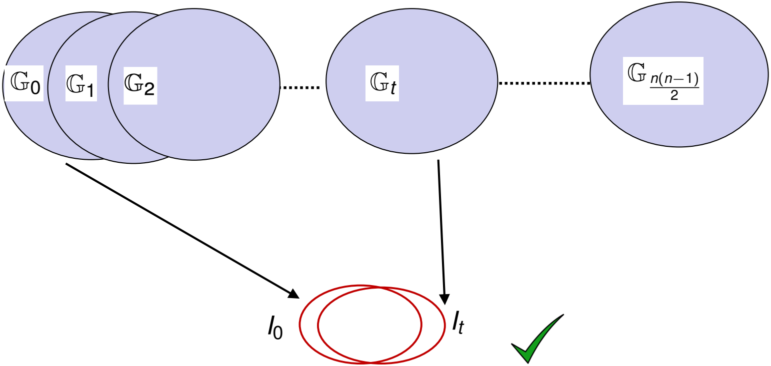

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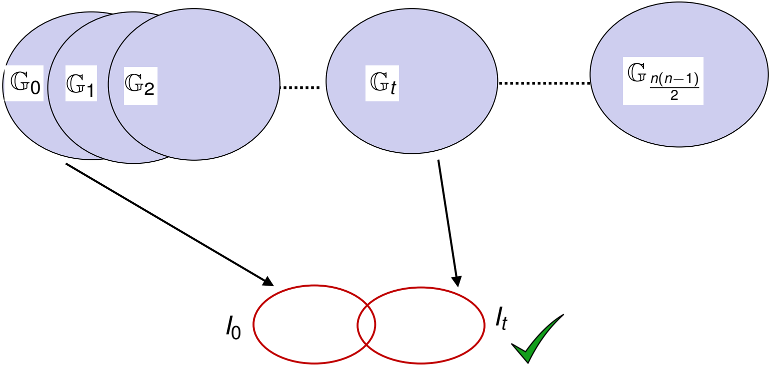

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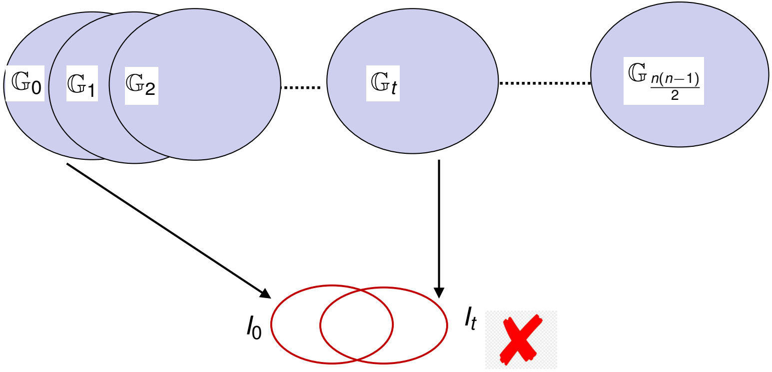

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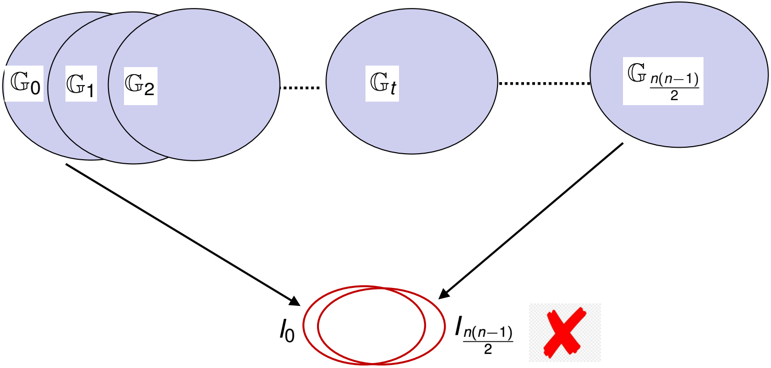

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Theorem (Wein 2020)

For every $\epsilon > 0$, $K \ge 1 + 5/\epsilon^2$ and d large enough the following holds with probability at least $1 - \exp(-\Omega(n))$: there does not exist a sequence of times $t_1, \dots, t_K$ with $0 \le t_k \le T$ and $1/2 + \epsilon$-optimal ind sets sets $I_1, \dots, I_K \subset [n]$ in $\mathbb{G}_{t_1}, \dots, \mathbb{G}_{t_K}$ such that

$$|I_k \setminus (\cup_{1 \le \ell < k} I_\ell)| \in \left[\frac{\epsilon}{4}\frac{\log d}{d} n, \frac{\epsilon}{2}\frac{\log d}{d} n\right], \qquad 2 \le k \le K.$$

e-OGP – obstruction to stable (noise-insensitive) algs

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Theorem (Meta-theorem)

Suppose an algorithm $\mathcal{A} : G \to \{0, 1\}^n$ is stable: a "small" change in input G results in a small change on the output $\mathcal{A}(G)$. Then $\mathcal{A}$ cannot overcome the e-OGP barrier.

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Theorem (Meta-theorem)

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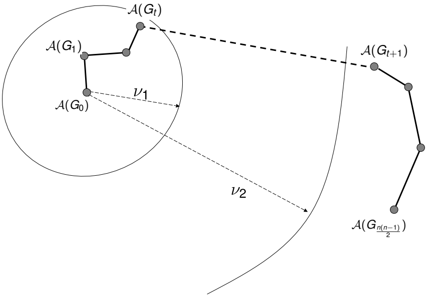
Proof by picture:



Boolean circuits. Background

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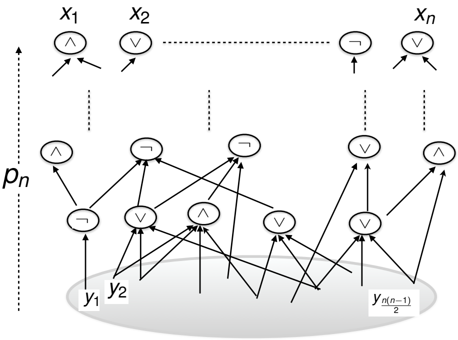
- Boolean circuit C – a mapping $\{0,1\}^{\binom{n}{2}} \to \{0,1\}^n$ encoded by a directed graph with logical gates $\neg, \vee, \wedge$
- Size $s(n)$– number of gates. Depth $p(n)$ – length of the longest path



Background: Circuit lower bounds

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$$\Omega\left(\log n/(k^2(n)\log\log n)\right) = o\left(\log n/\log\log n\right)$$

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Theorem (Circuit depth lower bound. G, Jagannath & Wein [2022])

Let $\alpha \in (1/2, 1), \epsilon > 0$ *and*

$$p(n) \leq \frac{\log n}{(1+\epsilon)\log\log n}.$$

Then for every $C \in \mathcal{C}(s(n), p(n), \alpha)$ *and all large enough n*

$$s(n) \geq n^{(\log n)^{\frac{\epsilon}{3}}}.$$

In particular, the size is super-polynomial.

Stability (noise insensitivity) of circuits: LMN Theorem

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Theorem (Linial-Mansour-Nisan' (LMN) Theorem, [1993])

For every circuit C with size $s(N)$ and depth $p(N)$ under the i.i.d. uniform distribution on $\{0,1\}^N$, the sum of Fourier coefficients associated with monomials of order

$$D_N \triangleq (\log s(N))^{O(p(N))} \qquad \left(\text{That is } (\log \text{Size})^{O(\text{Depth})}\right)$$

is $o(1)$.

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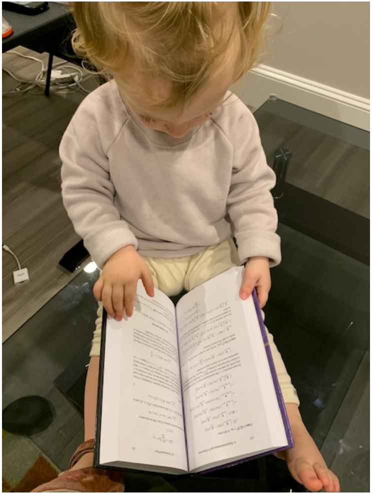
From O'Donnell "Analysis of Boolean Functions"



Stability (noise insensitivity) of circuits: Linial-Mansour-Nissan's Theorem

Informally, on $\{0,1\}^N$, the circuit C can be approximated by an N variable polynomial of degree $(\log \mathrm{Size})^{O(\mathrm{Depth})}$.

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Informally, on $\{0,1\}^N$, the circuit C can be approximated by an N variable polynomial of degree $(\log \mathrm{Size})^{O(\mathrm{Depth})}$.

Say, size $s(N) = N^\alpha$, depth $p(N) = \beta \log N/(\log\log N)$. Then

$$(\log \mathrm{Size})^{O(\mathrm{Depth})} \approx (\log N)^{\beta \log N/(\log \log N)} = \exp(\beta \log N) = N^\beta.$$

When $\beta < 1$ is small the "relevant" degree is sublinear in N.

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- A circuit C produces a sequence of solutions $I_t = C(G_t), t = 0, 1, \ldots, \binom{n}{2}$.

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- A circuit C produces a sequence of solutions $I_t = C(G_t), t = 0, 1, \ldots, \binom{n}{2}$.
- We use LMN and a large deviations estimate to show that

$$n^{-1}\|I_t - I_{t+1}\|_2^2 \leq (\nu_2 - \nu_1)^2,$$

for all t with probability *at least* $\exp(-\delta D_n)$ with controlled δ.

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$$\exp\left(-\delta(\log s(n))^{\frac{\log n}{(1+\epsilon)\log\log n}}\right) \leq \exp(-\Omega(n))$$

$$\implies$$

$$s(n) \geq n^{(\log n)^{\Omega(1)}} \quad \square$$

Small set avoidance

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Large deviation lower bound is based on the following lemma

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Lemma (Small Set Avoidance Lemma G, Jagannath & Wein [2020])

Let E be any subset of edges in $\{0,1\}^n$. Let e be the fraction of the edges in E. Consider a random walk $Z_t, 0 \leq t \leq T$ in $\{0,1\}^n$ with Z_0 chosen u.a.r. The walk never crosses the set E with probability at least 2^{-Te}.

OGP $\rightarrow$ spin glass hardness

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A very similar method shows failure of Boolean circuits in finding ground states of spin glass models https://arxiv.org/pdf/2109.01342.pdf and is likely to be applicable for all models exhibiting OGP.

Open questions

Challenges with applying this to other combinatorial optimization problems

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- Cliques on $\mathbb{G}(n, 1/2)$. The e-OGP occurs but with probability $1 - \exp(-\Theta(\log^2 n))$. This is too weak for obtaining circuit lower bounds.

Open questions

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- Cliques on $\mathbb{G}(n, 1/2)$. The e-OGP occurs but with probability $1 - \exp(-\Theta(\log^2 n))$. This is too weak for obtaining circuit lower bounds.
- Decision vs search lower bounds.
- Are there (evidently) hard problems not exhibiting OGP in the way we have defined it?