Equations and First-order Sentences in Random Groups

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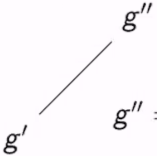
Random groups provide a rigorous way to tackle such questions as
“What does a typical (finitely generated) group look like?”
or
“What is the behavior of an element of a group when nothing particular happens?” We review the results

Gromov in 87: A typical group is hyperbolic.

Suppose $G = \langle x_1, \ldots, x_n | R \rangle$.

Cayley graph



$g'' = x_i^{\pm 1} g'$

G is (Gromov) hyperbolic if Cay(G) is hyperbolic, i.e. all geodesic triangles are thin

Definition (Few-relator model)

We fix $n \geq 2$. Let $\mathcal{G}_{k,\ell}$ be the following set of group presentations:

$$\{\langle e_1, \ldots, e_n \mid r_1, \ldots, r_k \rangle \ : \ r_i \text{ reduced } \& \ |r_i| \leq \ell\}$$

We say that property P, which is a property of presentations, happens almost surely in the few-relator model if the ratio of presentations in $\mathcal{G}_{k,\ell}$ which have property P over $|\mathcal{G}_{k,\ell}|$ goes to 1 as ℓ goes to infinity.

Gromov stated and Olshanskii proved that a random group in the few-relator model is almost surely hyperbolic.

Definition (Gromov's density model)

We fix $n \geq 2$. Let $\mathbb{F}_n := \langle e_1, \ldots, e_n \rangle$ be a free group of rank n. Let S_ℓ be the set of reduced words on $e_1, \ldots, e_n$ of length ℓ. Let $0 \leq d \leq 1$. Then a random set of relators of density d at length ℓ is a subset of S_ℓ that consists of $(2n-1)^{d\ell}$-many elements picked randomly (uniformly and independently) among all elements of S_ℓ.
A group $G := \langle e_1, \ldots, e_n \mid \mathcal{R} \rangle$ is called random of density d at length l if R is a random set of relators of density d at length ℓ.

A random group of density d almost surely satisfies some property of presentations P, if the probability of occurrence of P tends to 1 as ℓ goes to infinity.

Theorem (Gromov , Ollivier)

Let G be a random group of density d.

- *If $d < 1/2$. Then G is infinite torsion-free hyperbolic.*
- *if $d > 1/2$. Then G is either trivial or $\mathbb{Z}_2$.*

Theorem (Gromov , Ollivier)

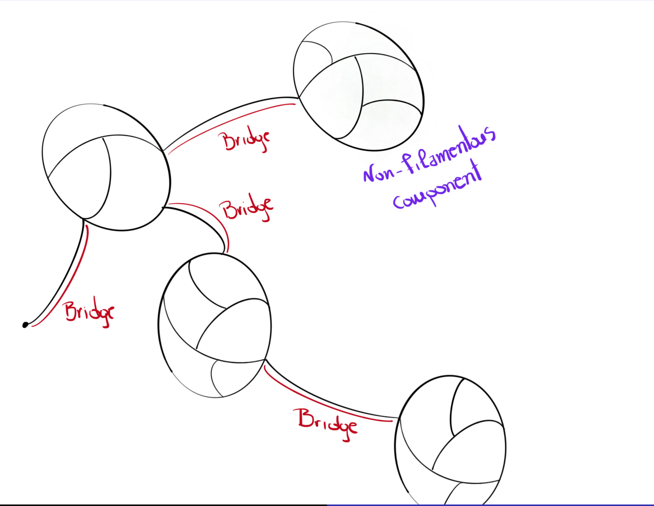
Let Γ_ℓ be a random group of density $d < 1/2$ at length ℓ. Every reduced diagram D over Γ_ℓ almost surely satisfies

$$\partial D > \ell(1-2d)|D|$$

where ∂D is the boundary length of D and $|D|$ is the number of its faces.

van Kampen diagrams and abstract vKd

Van Kampen diagrams are a visual way to represent how all equalities holding in a group are derived from combinations of relators.



A local to global. For each $\alpha > 0$ $\exists K(\alpha) > 1$ and $\alpha' > 0$ s.t. for a group given by relations of length ℓ
If $|\partial D| \geq \alpha\ell|D|$ for reduced vKd with at most $K(\alpha)$ faces,
Then $|\partial D| \geq \alpha'\ell|D|$ for any reduced vKd.

Probabilities. Let D be a reduced avKd, then either
$\partial D > \ell(1-2d)|D|$ or $Pr(D \text{ is fulfillable}) \leq (2n-1)^{-\ell(1/2-d)/2}$.

Small Cancellation

Let

$$G = \langle X \mid R \rangle \qquad (*)$$

be a group presentation where $R \subseteq F(X)$ is a set of freely reduced and cyclically reduced words in the free group $F(X)$ such that R is *symmetrized*, that is, closed under taking cyclic permutations and inverses.

A nontrivial freely reduced word u in $F(X)$ is called a *piece* with respect to $(*)$ if there exist two distinct elements r_1, r_2 in R that have u as maximal common initial segment.

Note that if $G = \langle X \mid S \rangle$ is a group presentation where the set of defining relators S is not symmetrized, we can always take the *symmetrized closure* R of S, where R consists of all cyclic permutations of elements of S and S^{-1}. Then R is symmetrized and $G = \langle X \mid R \rangle$ is also a presentation of G.

Let $0 < \lambda < 1$. Presentation $(*)$ as above is said to satisfy the $C'(\lambda)$ *small cancellation condition* if whenever u is a piece with respect to $(*)$ and u is a subword of some $r \in R$, then $|u| < \lambda |r|$. Here $|v|$ is the length of a word v.

Small cancellation

Theorem (Gromov)

Let $\alpha > 0$ and $d < \alpha/2$, then WOP a random set of relators at density d satisfies $C'(\alpha)$.

Theorem (Callegari-Walker)

Let $d < 1/2$. Then a random group at length ℓ contains a surface group with probability $1 - O(e^{-\ell})$.

Tarski in 1946 asked whether the first-order theories (true sentences) of nonabelian free groups of different ranks are the same. This question was answered in the positive in 2006, by Sela and Kharlampovich-Myasnikov.
Algebraic geometry in a free group was developed (Baumslag, Remeslennikov, Miasnikov, Kh, and later Amaglobeli, Danijarova etc)

Conjecture[J. Knight]
Let σ be a first-order sentence in the language of groups. Then σ is true in a nonabelian free group if and only if it is almost surely true in "the random group". (She, actually, means random one relator group.)

Main results

Let F_n be a free group of rank n and Γ_ℓ a random group with relations of length ℓ. Let $V(\bar{x}) = 1$ be a system of equations.

$$
\begin{tikzcd}
G_V = \langle \bar{x} \mid V(\bar{x}) \rangle \arrow[r, "h"] \arrow[d, dashed] & \Gamma_\ell = \langle e_1, \dots, e_n \mid R \rangle \\
F_n \arrow[ur, "\pi_\ell"'] &
\end{tikzcd}
$$

Such h is called a free sol'n of $V(\bar{x}) = 1$

Figure: free solution of $V(\bar{x}) = 1$ in Γ_ℓ

Main results

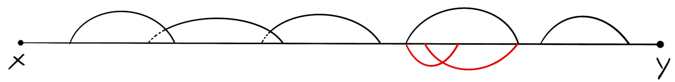
Theorem (Kh-Sklinos)

Let the density $d < 1/16$, then $Pr(\exists h\ non-free) \to 0$ as $\ell \to \infty$.

Theorem (Kh-Sklinos)

Let the density $d < 1/16$ and σ be a $\forall\exists$-sentence. Then $Pr(F_n \models \sigma \text{ while } \Gamma_\ell \models \neg\sigma) \to 0$ as $\ell \to \infty$.

Ideas Local to global: Shortening argument. Strebel classification of geodesic triangles in small cancellation groups; Single layer configuration of geodesics in $C'(1/8)$ groups.



Local to global for equations

Definition

Let $G := \langle g_1, g_2, \ldots, g_m \rangle$ be a finitely generated group. Let $h : G \to H$ be a morphism to a finitely generated group H. Then the length of h, denoted $|h|$, is the sum $|h(g_1)| + |h(g_2)| + \ldots + |h(g_m)|$.

Theorem

We fix $d < 1/16$ and a system of equations $V(\bar{x}) = 1$. Let $(\Gamma_i)_{i<\omega}$ be a sequence of random groups of density d at length i and $(h_i)_{i<\omega} : G_V \to \Gamma_i$ a sequence of short non free morphisms. Then, there is a constant K (that depends only on $V(\bar{x}) = 1$ and d) such that $|h_i| \leq Ki$.

This bounds by another constant K_1 the number of faces in the family of diagrams we have to consider for each ℓ. This number of families of diagrams for a given ℓ growth polynomially with ℓ.

Ideas in the proof of the existence of K

Let Γ_i be a group with $C'(1/8)$ with defining relators of length i. Suppose such constant K does not exist. We consider a (stable) sequence of shortest non-free morphisms $(h_i)_{i<\omega} : G \to \Gamma_i$ from a finitely generated group G such that $|h_i|$ dominates i, i.e. $\frac{i}{|h_i|}$ goes to 0 as i goes to infinity.
We call a group obtained as a quotient, $G/\underrightarrow{Ker}(h_i)$, *a $\mathcal{G}_8$-limit group* and the canonical map $\varphi : G \to L := G/\underrightarrow{Ker}(h_i)$ *the limit map*. It is known that this limit group acts in an $\mathbb{R}$-tree.

Recall, that an action on an $\mathbb{R}$-tree is *superstable* if whenever, $J \subseteq I$ for segments I, J, we have $Fix(J) \neq Fix(I)$, where $Fix(I)$ denotes the pointwise stabilizer of I, then $Fix(I) = 1$.

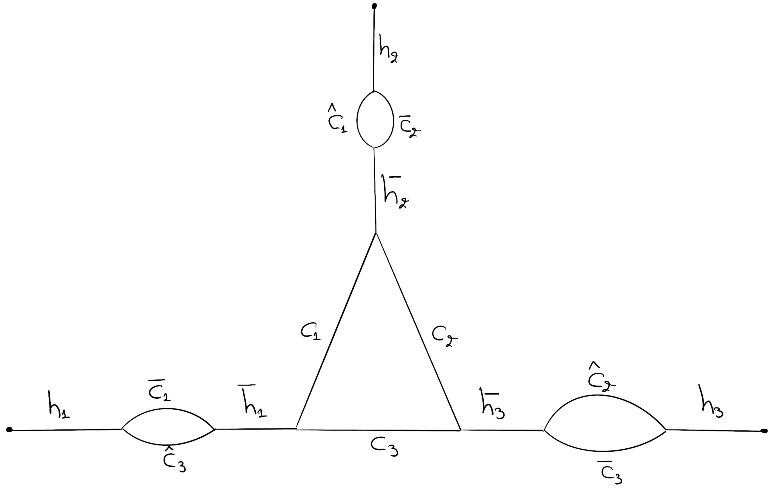
Theorem

If $\{\Gamma_i\}$ is a family of torsion free eighth groups, then the action of the limit group $G/\underrightarrow{Ker}(h_i)$ on the $\mathbb{R}$-tree T is superstable.

The existence of such action implies that homomorphisms can be shorten, therefore contradicts the assumption that no K exists.

Fulfilling a given family of diagrams

We can assume assume the diagrams representing a solution of the system of equations $V(x_1, \ldots, x_m) = 1$ are triangles



Fulfilling a given family of diagrams

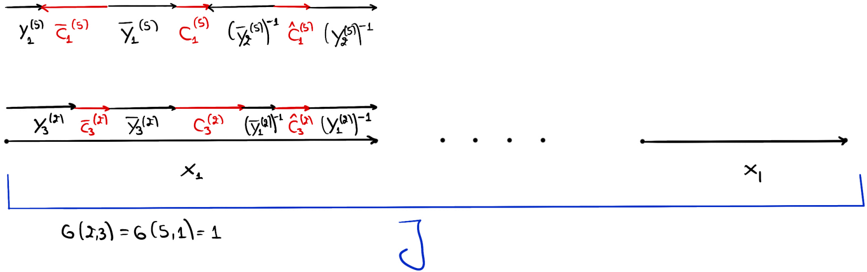


Figure: System $\Sigma(\bar{y}, \bar{c}) = 1$, where the parameters $\bar{c}$ are in red color.

Fulfilling a given family of non-filamentous diagrams

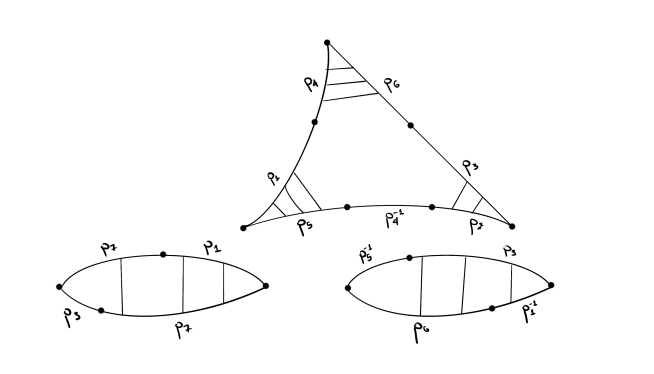


Figure: Decoration on the boundaries

Fulfilling a given family of non-filamentous diagrams

Suppose the decoration P on the boundary of a non-filamentous aDVK D was imposed by a non-free solution of $V(\bar{x}) = 1$. Let Γ_ℓ be a group with $(2n-1)^{d\ell}$ random relations of length ℓ.

Lemma

The probability that D can be labelled by relations of Γ_ℓ such that the labeling satisfies P is not more than

$$(2n-1)^{-\ell(1/2-d)}.$$

Theorem (Dahmani-Guirardel-Przytycki)

Let $0 < d < 1$. A random group at density d has almost surely property (FA).

Theorem

If $0 < d \leq 1/2$, then a random group is not a limit group.

$\forall$-sentences

The random group of fixed density $d < 1/2$ does not satisfy a universal sentence which is not in the axioms of the theory of the free group.
Indeed, the random group contains a nonabelian free group, hence it satisfies the existential theory of the free group.

Consider a universal axiom σ in T_{fg} of the form

$$\forall \bar{x}(w_1(\bar{x}) = 1 \vee w_2(\bar{x}) = 1 \vee w_k(\bar{x}) = 1 \vee v_1(\bar{x}) \neq 1 \ldots \vee \ldots \vee v_m(\bar{x}) \neq 1).$$

Let $V(\bar{x}) = (v_1(\bar{x}) = 1 \wedge \ldots v_m(\bar{x}) = 1)$. This is equivalent to

$$\forall \bar{x}(V(\bar{x}) = 1 \to w_1(\bar{x}) = 1 \vee \ldots \vee w_k(\bar{x}) = 1)$$

Now, Γ_ℓ satisfies $\neg\sigma$, while $\mathbb{F}_n$ satisfies σ, only if there exists a tuple $\bar{b}$ in Γ_ℓ such that $V(\bar{b}) = 1$, while for any pre-image $\bar{c}$ of $\bar{b}$ via $\pi : F_n \to \Gamma_\ell$ we have $\vee_{i=1}^{m} v_i(\bar{c}) \neq 1$ in $\mathbb{F}_n$.