On compromise solutions and scalarization in multiobjective optimization

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Consider a general multiobjective optimization problem (MOP) as: $\min_{\mathbf{x}\in X} f(\mathbf{x})$. The feasible set is $X \subseteq R^n$ and $f : R^n \longrightarrow R^m$ is the objective function. The natural ordering cone is defined as $R^m_{\geqq} = \{\mathbf{x} \in R^m \ : \ x_j \geq 0, \ \forall j = 1, 2, \ldots, m\}$. A feasible solution $\hat{\mathbf{x}} \in X$ is called an efficient solution to (MOP) if $(f(\hat{\mathbf{x}}) - R^m_{\geqq}) \bigcap f(X) = \{f(\hat{\mathbf{x}})\}$.

Definition 1. [4] An efficient solution $\hat{\mathbf{x}} \in X$ is called a properly efficient solution to (MOP), if there is a real number $M > 0$ such that for all $i \in \{1, 2, ..., m\}$ and $\mathbf{x} \in X$ with $f_i(\mathbf{x}) < f_i(\hat{\mathbf{x}})$ there exists $j \in \{1, 2, ..., m\}$ such that $f_j(\mathbf{x}) > f_j(\hat{\mathbf{x}})$ and $\frac{f_i(\hat{\mathbf{x}}) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\hat{\mathbf{x}})} \leq M$.

Definition 2. [2] The point $\mathbf{y}^I = (y_1^I, \ldots, y_m^I)$ in which $y_i^I = min_{\mathbf{x}\in X} f_i(\mathbf{x})$, is said the ideal point of (MOP). The point $\mathbf{y^U} = \mathbf{y^I} - \boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} > \mathbf{0}$, is called an utopia point.

One of the popular measure functions, which has been widely used in the literature, is defined by $d(\boldsymbol{\lambda}, \mathbf{y}) = \|\boldsymbol{\lambda} \odot \mathbf{y}\|_p$ for each $(\boldsymbol{\lambda}, \mathbf{y}) \in R^m \times R^m$, in which p is a positive integer, $\boldsymbol{\lambda} \odot \mathbf{y} = (\lambda_1 y_1, \lambda_2 y_2, \ldots, \lambda_m y_m)$ and $\|\boldsymbol{\lambda} \odot \mathbf{y}\|_p = \left(\sum_{j=1}^{m} |\lambda_j y_j|^p\right)^{\frac{1}{p}}$. Considering a vector $\boldsymbol{\lambda} > 0$, the set of best approximations of the ideal point measured by $\|.\|_p$ is defined by $A(\boldsymbol{\lambda}, p, Y) = \left\{\bar{\mathbf{y}} \in Y \ : \ \|\boldsymbol{\lambda} \odot (\bar{\mathbf{y}} - \mathbf{y}^U)\|_p = \min_{\mathbf{y}\in Y} \|\boldsymbol{\lambda} \odot (\mathbf{y} - \mathbf{y}^U)\|_p\right\}$, in which $Y = f(X)$. Now, the set of best approximations of $\mathbf{y}^I$ considering all positive weights (compromise solutions) is defined by $A(Y) = \bigcup_{\boldsymbol{\lambda}\in\Lambda^0} \bigcup_{1\leq p<\infty} A(\boldsymbol{\lambda}, p, Y)$ where $\Lambda^0 = \{\boldsymbol{\lambda} \in R^m \ : \ \sum_{j=1}^m \lambda_j = 1, \ \lambda_j > 0, \ \forall j = 1, 2, \ldots, m\}$.

In this talk, some important connections between the members of $A(Y)$ and proper efficient solutions (without $R^m_{\geqq}-$closedness) are discussed. In the second part of the talk, some scalarization problems are considered. In the literature it was proved that, under convexity assumption, the set of properly efficient points is empty when some scalarization problem(s) is (are) unbounded. In this talk, some connections between the proper efficient solutions, compormise solutions and the solutions of scalarization problems are discussed.

References

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