Multivariable Nevanlinna-Pick interpolation: the free noncommutative setting

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We are now approaching the centennial of the discovery of Nevanlinna (1919) and Pick (1916) of what is now called **Nevanlinna-Pick interpolation**: *given a finite collection of points $z_1, \dots, z_N$ in the unit disk $\mathbb{D}$ and a corresponding finite collection of prescribed values $w_1, \dots, w_N$ in $\mathbb{C}$, then there is a holomorphic function f mapping the unit disk $\mathbb{D}$ into the closed disk $\overline{\mathbb{D}}$ satisfying the prescribed interpolation conditions $f(z_i) = w_i$ if and only if the associated Pick matrix $\mathbb{P} = \left[\frac{1-\lambda_i\overline{\lambda_j}}{1-z_i\overline{z_j}}\right]$ is positive semidefinite.* Over the years there has now been a lot of developments extending the theory of Nevanlinna-Pick interpolation to more general target spaces (matrix- or operator-valued instead of scalar-valued—significantly inspired by connections with the emerging H^∞-control theory) as well as to more general domains (more general planar as well as multivariable domains in place of the unit disk). Traditionally these multivariable domains involve only commuting variables.

It is only recently that a new "free analysis", or the study of "noncommutative functions" has emerged, where the domain consists of freely noncommuting variables into which one plugs matrix- or even operator-valued arguments. A **noncommutative function** is defined to be a function of freely noncommuting square-matrix arguments (of arbitrary size) which is required to have certain natural invariance properties with respect to direct sums and similarity transforms; these axioms combined with weak boundedness assumptions lead to strong analyticity properties. A number of groups of researchers have come upon this class of functions for completely independent reasons (development of a meaningful functional calculus for noncommuting operator tuples, the theory of formal languages and free automata, free probability, dimension-less optimization problems in systems engineering, input/state/output linear systems with evolution along a free monoid). We refer to the recent monograph [1] for more complete details on both the historical context and the basic theory.

In this talk we discuss a free version of the Nevanlinna-Pick interpolation problem: *given prescribed interpolation nodes $Z^{(1)}, \dots, Z^{(N)}$ in a noncommutative domain $\mathbb{D}_Q$ and associated matrix values $\Lambda_1, \dots, \Lambda_N$, find a contractive noncommutative function S on $\mathbb{D}_Q$ so that $S(Z^{(i)}) = \Lambda_i$ for $i = 1, \dots, N$.* We emphasize both the similarities with and the points of departure from the earlier (commutative-variable) versions of Nevanlinna-Pick interpolation. This is joint work with Gregory Marx (Virginia Tech) and Victor Vinnikov (Ben Gurion University).

References:

[1] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of Noncommutative Function Theory*, Mathematical Surveys and Monographs **199**, Amer. Math. Soc., Providence, 2014.